Math 1511 G Final Review

Last Edited: Spring 2022

1. Evaluate the following limits using the method of your choice. Show work to justify your answer. If the limit does not exist, state it clearly.

a. $\lim_{x\to -1} 2x^2$

b. $\lim_{x\to 9} \frac{3-\sqrt{x}}{9-x}$

c. $\lim_{x\to -\infty} \frac{x^2-3x+1}{2x^3+4x-6}$

d. $\lim_{x\to -1} \frac{\tan x}{\ln x}$

e. $\lim_{x\to 5} \frac{x^2-6x+5}{x-5}$

f. $\lim_{x\to 1} \frac{e^{-x}-\ln(x)}{4x-1}$

g. $\lim_{x\to 0} \frac{e^{x+1}-e}{xe^x}$

h. $\lim_{x\to 0^-} \frac{\sin x}{\cos x - 1}$

i. $\lim_{x\to 0} \frac{3^x-1}{(\ln 3)\cdot x}$

j. $\lim_{x\to \infty} \frac{3x^2+2x-1}{x^2-1}$

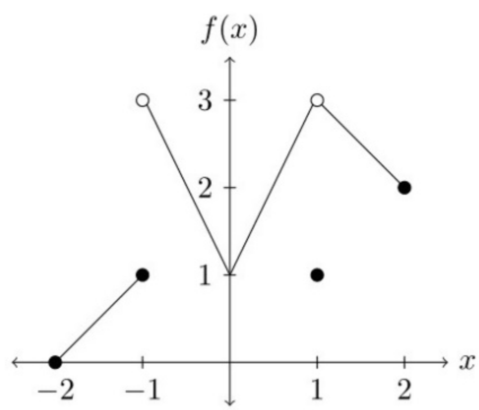
k. $\lim_{y\to 0^+} \frac{\ln(y^2+2y)}{\ln y}$

l. $\lim_{x\to \infty} \frac{x+2x^2}{\sqrt{x^4-1}}$

m. $\lim_{x\to -3} \frac{(x+3)\sqrt{2x^2+5}}{x^2-9}$

n. $\lim_{x\to 0} \frac{3x^2}{\pi - \sin(x)}$

2. Evaluate or answer the following based on the graph provided.



a.

i. $f(-1)$

ii. $f(1)$

iii. $\lim_{x\to -1} f(x)$

iv. $\lim_{x\to 1} f(x)$

v. $\lim_{x\to -1^+} f(x)$

vi. $\lim_{x\to 0} f(x)$

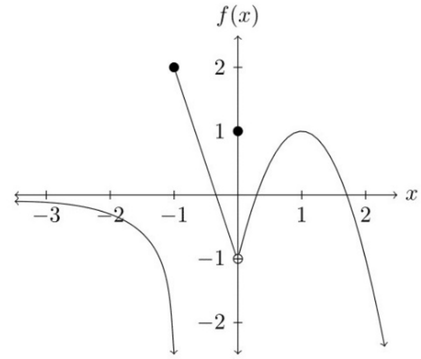
vii. $f(0)$

viii. Give the intervals over which $f(x)$ is continuous. For each discontinuity, state the type of discontinuity.

ix. For which x-values is $f(x)$ NOT differentiable? State the reason for each.

b.

i. $f(-1)$
ii. $f(0)$
iii. $\lim_{x\to -1} f(x)$
iv. $\lim_{x\to\infty} f(x)$
v. $\lim_{x\to-\infty} f(x)$
vi. $\lim_{x\to 0} f(x)$
vii. $\lim_{x\to -1^-} f(x)$



viii. Estimate $f(2)$ from the graph.
ix. Give the intervals over which $f(x)$ is continuous. For each discontinuity, state the type of discontinuity.
x. For which x-values is $f(x)$ NOT differentiable? State the reason for each.
xi. State the number of x-values for which $f(x) = 0$.
xii. State the number of x-values for which $f'(x) = 0$.
xiii. State the number of x-values for which $f'(x) = -1$.

3. If possible, sketch a graph for a function that satisfies the following conditions.

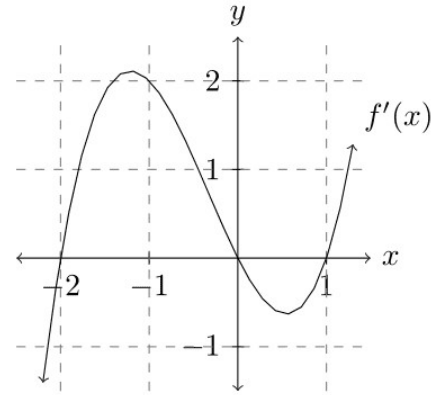
a. $\lim_{x\to-\infty} g(x) = 0$
$g(-1) = 2$
$\lim_{x\to-2^-} g(x) = -\infty$
$\lim_{x\to-2^+} g(x) = -\infty$
$g'(-1) = 0$
$g(1) = 0$
$\lim_{x\to1} g(x) = -1$
$\lim_{x\to\infty} g(x) = -\infty$

b. $\lim_{x\to-\infty} f(x) = \infty$
$f(x)$ is continuous but not differentiable at $x = -1$
$f(0) = 1$
$\lim_{x\to0^+} f(x) = 2$
$\lim_{x\to0^-} f(x) = 0$
$f'(1) = -1$
$f(2) = 0$
$\lim_{x\to\infty} f(x) = 0$

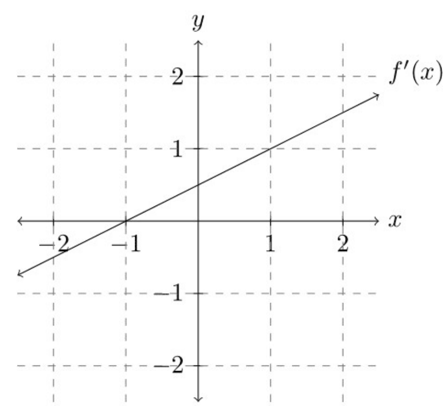
4. Let $f(x)$ be differentiable at $x = a$.
    a. State the limit definition of $f'(a)$.
    b. Give two important interpretations of $f'(a)$.
    c. Using the definition in (a), find $f'(2)$ for $f(x) = \sqrt{2x+1}$.
    d. Using the definition in (a), find $f'(4)$ for $f(x) = \frac{1}{x+1}$.
    e. Using the definition in (a), find $f'(1)$ for $f(x) = 5x^2 - x$.

5. Suppose that $f(2+h) - f(2) = 2h^2 + 7h$. Calculate $f'(2)$.

6. For the following problems, the graph of the derivative of a function $f(x)$ is given. Answer the questions based on the graph.
    a.
        i. State the critical points of the function $f(x)$ and determine whether each represents a minimum, maximum, or neither.
        ii. Give the intervals of increase and decrease for $f(x)$.
        iii. Estimate the interval(s) over which $f(x)$ is concave up and concave down.
        iv. If $f(-1) = 4$, find the equation of the tangent line to $f(x)$ at $x = -1$.
        v. If $f(1) = 2$, find the equation of the tangent line to $f(x)$ at $x = 1$.



    b.
        i. State the critical points of the function $f(x)$ and determine whether each represents a minimum, maximum, or neither.
        ii. Give the intervals of increase and decrease for $f(x)$.
        iii. Give the interval(s) over which $f(x)$ is concave up and concave down.
        iv. If $f(-1) = 4$, find the equation of the tangent line to $f(x)$ at $x = -1$.
        v. If $f(1) = -2$, find the equation of the tangent line to $f(x)$ at $x = 1$.



7. Find the derivative with respect to the independent variable. Use notation consistent with the initial function. You must simplify to receive full credit.

a. $g(z) = \dfrac{1 - e^z}{z^3}$

b. $h(t) = \cos(\ln(1 + t^2))$

c. $y = \sin x\,[1 - \ln(\sin x)]$

d. $g(y) = 3e^y - \cos y + 4^y - \tan^{-1} y - e^\pi$

e. $g(t) = \dfrac{\cos(e^{-t})}{e^t}$

f. $f(\theta) = \sin(\cos(2\pi\theta))$

g. $g(x) = 4x^2 \ln(\sqrt{x})$

h. $y = -2\cos^{-1}(3x - 4)$

i. $h(z) = 6z^{\frac{5}{3}} - \dfrac{4}{z^2} + 7\sqrt{z} - 3\log_2 z$

j. $h(t) = 2^{t^2 - t}$

8. For each of the following, use implicit differentiation to find $\frac{dy}{dx}$ and then find the equation of the tangent line to the curve at the given point.
   a. $x^2 + \sin y = xy^2 + 1 \quad , \quad (1, 0)$
   b. $xe^y + ye^x = 1 \quad , \quad (0, 1)$

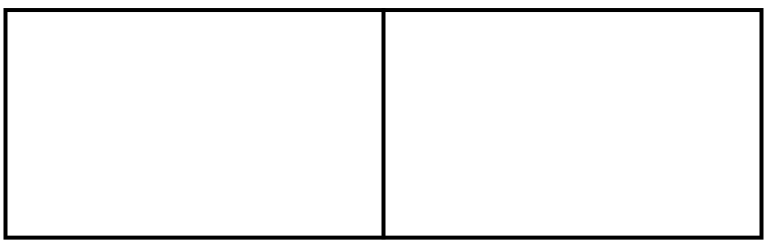
9. Use the hyperbola, $9x^2 - y^2 = 36$, to answer each question.
   a. Verify that $\left(\frac{10}{3}, 8\right)$ is a point on the hyperbola.
   b. Write the equation of the tangent line to the hyperbola at the point $\left(\frac{10}{3}, 8\right)$. Write the equation in the slope-intercept form. Write values either as integers or fractions in simplest form.
   c. Find the points on the hyperbola where the tangent line is vertical.

10. For $x^2 + y^3 = y + x$
    a. Find ${}^{dy}\!/_{dx}$
    b. Find ${}^{d^2y}\!/_{dx^2}$
    c. Find the equation of the tangent line through the origin.

11. Let $f(x) = 16 + 2x^2 - x^4$ on the interval $[-2,2]$.
    a. Find the first and second derivatives of f.
    b. Find the critical points of f. Evaluate f at the critical points and the end points. Give the coordinates of any global extrema.
    c. Find points of inflection, indicate where the function is concave up or down.
    d. Sketch a graph of $f(x)$ on the given interval.

12. Let $f(x) = \sin x - \cos x$ on the interval $[0, \pi]$.
    a. Find the first and second derivatives of f.
    b. Find the critical points of f. Evaluate f at the critical points and the end points. Give the coordinates of any global extrema.
    c. Find points of inflection, indicate where the function is concave up or down.
    d. Sketch a graph of $f(x)$ on the given interval.
13. Let $f(x) = \sqrt{x}$. Use a linearization of $f(x)$ to approximate $\sqrt{120}$.
14. Let $f(x) = e^x$. Use a linearization of $f(x)$ to approximate $e^{0.1}$.
15. Let $f(x) = x^2 - x + 1$. Show that $f(x)$ satisfies the conditions of the Mean Value Theorem on the interval $[0, 2]$ and find the value of c.
16. It took 16 sec for the temperature to rise from $0°\text{F}$ to $212°\text{F}$ when a thermometer was taken from a freezer and placed in boiling water. Explain why at some moment in that interval the thermometer's mercury was rising at exactly $13.25°\text{F/sec}$.
17. A rock is dropped from a point on a bridge which is 50 meters above the water. Its height above the water as a function of time is $h(t) = 50 - 4.9t^2$ meters. What is the average velocity of the rock for the first 2 seconds of its fall? What is the instantaneous velocity of the rock at the end of that time?
18. Gravel is being dumped from a conveyor belt at a rate of 30 ft^3/min. The gravel forms a pile in the shape of a cone whose height is always twice its radius. How fast is the height of the pile changing when the pile is 10 feet high? Include correct units. Show work.
19. Two cars are moving away from an intersection along different roads that make a 90° angle. The first car is traveling at a rate of 30 ft/s and the other is traveling at a rate of 40 ft/s. At the moment that the first car is 50 ft from the intersection and the other car is 120 ft from the intersection, what is the rate of change of the distance between the two cars?
20. A 13 foot ladder is leaning against a wall when its base starts to slide away from the wall. By the time the base is 12 ft. from the wall, the base is moving at the rate of 5 ft./sec. How fast is the ladder sliding down the wall?
21. A conical reservoir is 100 feet in diameter at the top and 20 feet deep in the center. Water is flowing into the reservoir at 200 cubic feet per minute. At what rate is the depth changing when the water level is 12 feet below the top?
22. A rocket travels vertically at a speed of 1200 km/h. The rocket is tracked through a telescope by an observer located 15 km from the launching pad. Find the rate at which the angle between the telescope and the ground is increasing 3 min after lift-off.

23. A cylindrical canister of volume 0.1 cubic meters is constructed out of two materials. The top and bottom are constructed out of plastic that costs \$3 per square meter. The sides are constructed out of cardboard that costs \$1 per square meter. Find the exact radius that will minimize the cost of producing the canister.
24. According to postal regulations, a carton is classified as “oversized” if the sum of its height and girth (perimeter of its base) exceeds 108 in. Find the dimensions of a carton with a square base that is not oversized and has a maximum volume.
25. Find the area of the largest isosceles triangle that can be inscribed in a circle of radius 1.
26. You want to fence in a rectangular garden to take up 150 ft^2 of space in your backyard. You also want to divide the rectangular space into 2 equal sections with the fencing so you can grow different vegetables (see figure below).
    a. What is the minimum amount of fencing needed to build the garden? Answer = 60 ft
    b. What will the overall dimensions of the garden be? Answer = 10ft x 15ft



27. The given functions describe the position of an object in meters after t seconds. Answer the following for each function.
    i. Calculate the average velocity of the object over the time interval $[1, 4]$.
    ii. Find the velocity and acceleration functions.
    iii. Calculate the instantaneous velocity at $t = 4$ seconds.
    iv. Determine the interval(s) over which the object is moving in a positive direction and the intervals over which the object is moving in a negative direction.
    v. Determine the intervals over which the object is speeding up, slowing down.

    a. $s(t) = -3t^2 + 6t + 3$
    b. $s(t) = t^3 - 6t^2 + 9t + 3$
28. Find the antiderivative of each function.
    a. $f(x) = 2x^3 - \frac{1}{3}e^x + 3 + e$
    b. $g(x) = 7x^5 + 2x + \frac{1}{x} - e^x$

Answer Key

1.

a. $\lim_{x\to -1} 2x^2 = 2$

b. $\lim_{x\to 9} \frac{3-\sqrt{x}}{9-x} = \frac{1}{6}$

c. $\lim_{x\to -\infty} \frac{x^2-3x+1}{2x^3+4x-6} = 0$

d. $\lim_{x\to -1} \frac{\tan x}{\ln x}$ does not exist (DNE)

e. $\lim_{x\to 5} \frac{x^2-6x+5}{x-5} = 4$

f. $\lim_{x\to 1} \frac{e^{-x}-\ln(x)}{4x-1} = \frac{1}{3e}$

g. $\lim_{x\to 0} \frac{e^{x+1}-e}{xe^x} = e$

h. $\lim_{x\to 0^-} \frac{\sin x}{\cos x - 1} = \infty$

i. $\lim_{x\to 0} \frac{3^x-1}{(\ln 3)\cdot x} = 1$

j. $\lim_{x\to \infty} \frac{3x^2+2x-1}{x^2-1} = 3$

k. $\lim_{y\to 0^+} \frac{\ln(y^2+2y)}{\ln y} = 1$

l. $\lim_{x\to \infty} \frac{x+2x^2}{\sqrt{x^4-1}} = 2$

m. $\lim_{x\to -3} \frac{(x+3)\sqrt{2x^2+5}}{x^2-9} = -\frac{\sqrt{23}}{6}$

n. $\lim_{x\to 0} \frac{3x^2}{\pi - \sin(x)} = 0$

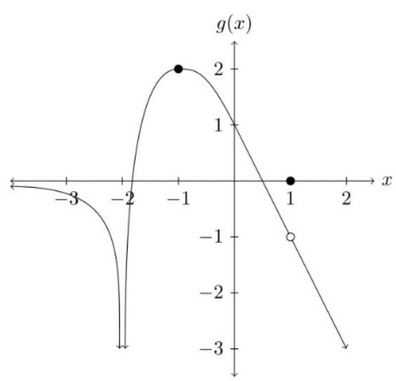
2.

a.
- i. 1
- ii. 1
- iii. DNE
- iv. 3
- v. 3
- vi. 1
- vii. 1
- viii. $[-2,-1),(-1,1),(1,2]$, jump discontinuity at $x=-1$, removable discontinuity at $x=1$.
- ix. $x=\pm 1$ both due to discontinuity, $x=0$ due to a cusp

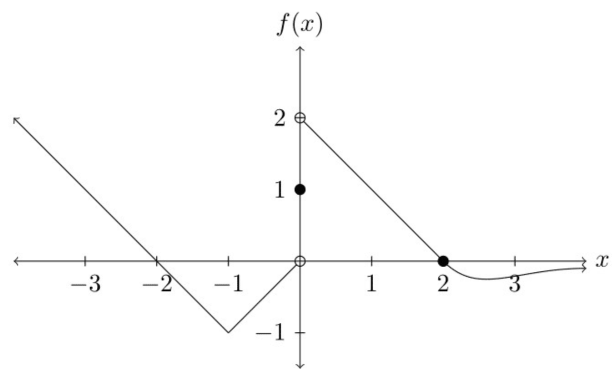
b. i. 2
ii. 1
iii. DNE
iv. $-\infty$
v. 0
vi. -1
vii. $-\infty$
viii. -1
ix. $(-\infty,-1),(-1,0),(0,\infty)$, infinite discontinuity at $x=-1$, removable discontinuity at $x=0$.
x. $x=-1,0$ both due to discontinuity.
xi. 3
xii. 1
xiii. 2, once on the interval (-2,-1) and once on the interval (1,2).

3. Answers will vary. These are possible sketches.

a.



b.



4.

a. $\lim\limits_{x\to a}\dfrac{f(x)-f(a)}{x-a}$ or $\lim\limits_{h\to 0}\dfrac{f(a+h)-f(a)}{h}$

b. Instantaneous rate of change at $x=a$; slope of tangent line to $f(x)$ at $(a,f(a))$.

c. $\lim\limits_{x\to 2}\dfrac{\sqrt{2x+1}-\sqrt{5}}{x-2}=\dfrac{1}{\sqrt{5}}$

d. $\lim\limits_{x\to 4}\dfrac{\frac{1}{\sqrt{x}}-\frac{1}{2}}{x-4}=-\dfrac{1}{16}$

e. $\lim\limits_{x\to 1}\dfrac{5x^2-x-4}{x-1}=9$

5. $\lim\limits_{h\to 0}\frac{2h^2+7h}{h}=7$

6. a. i. -2 (min), 0 (max), 1 (min)

ii. Increasing $(-2,0)$ and $(1,\infty)$, decreasing $(\infty,-2)$ and $(0,1)$

iii. Concave up $(-\infty,-1.2)$ and $(0.6,\infty)$, concave down $(-1.2,0.6)$

iv. $y = 2x + 6$

v. $y = 2$

b. i. -1 (min)

ii. Increasing $(-1,\infty)$, decreasing $(-\infty,-1)$.

iii. Concave up $(-\infty,\infty)$

iv. $y = 4$

v. $y = x - 3$

7.

a. $g'(z) = \dfrac{(3-z)e^z - 3}{z^4}$

b. $h'(t) = -\dfrac{2t\sin(\ln(1+t^2))}{1+t^2}$

c. $y' = -\cos x \cdot \ln(\sin x)$

d. $g'(y) = 3e^y + \sin y + 4^y \ln 4 - \dfrac{1}{1+y^2}$

e. $g'(t) = e^{-2t}\sin(e^{-t}) - e^{-t}\cos(e^{-t})$

or

$g'(t) = \frac{\sin(e^{-t}) - e^t\cos(e^{-t})}{e^{2t}}$

f. $f'(\theta) = -2\pi\sin(2\pi\theta)\cos(\cos(2\pi\theta))$

g. $g'(x) = 8x\ln(\sqrt{x}) + 2x$

h. $y' = \dfrac{6}{\sqrt{1-(3x-4)^2}}$

i. $h'(z) = 10z^{2/3} + \dfrac{8}{z^3} + \dfrac{7}{2\sqrt{z}} - \dfrac{3}{z\ln 2}$

j. $h'(t) = \ln 2\left(2^{t^2-t}\right)(2t-1)$

8.

a. $\frac{dy}{dx} = \frac{y^2-2x}{\cos y - 2xy}, \left.\frac{dy}{dx}\right|_{(1,0)} = -2, \text{tangent: } y = -2x + 2$

b. $\frac{dy}{dx} = -\frac{ye^x + 2xe^y}{x^2e^y + e^x}, \left.\frac{dy}{dx}\right|_{(2,0)} = -\frac{4}{4+e^2}, \text{tangent: } y = -\frac{4x}{4+e^2} + \frac{8}{4+e^2}$

9.

b. $y = \dfrac{15}{4}x - \dfrac{9}{2}$

c. $(-2,0)$ and $(2,0)$

10.

a. $\dfrac{dy}{dx} = \dfrac{1-2x}{3y^2-1}$

b. $\dfrac{d^2y}{dx^2} = \dfrac{-2(3y^2-1)^2-(1-2x)^2(6y)}{(3y^2-1)^3}$

c. $y = -x$

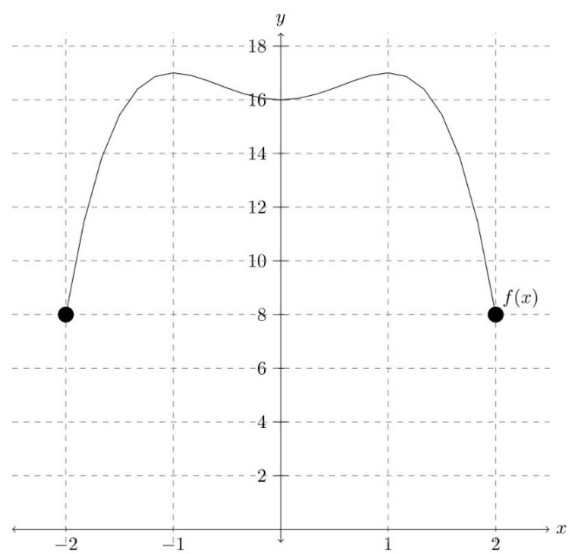
11.

a. $f'(x) = 4x - 4x^3$; $f''(x) = 4 - 12x^2$

b. Critical points: $x = 0, \pm 1$. $f(0) = 16, f(\pm 1) = 17, f(\pm 2) = 8$. Global maxima: $(-1,17), (1,17)$. Global minima: $(-2,8), (2,8)$.

c. Inflection points: $\left(\pm\frac{1}{\sqrt{3}}, \frac{149}{9}\right)$. Concave up on $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, concave down on $\left(-2, -\frac{1}{\sqrt{3}}\right)$ and $\left(\frac{1}{\sqrt{3}}, 2\right)$.

d. Sketch:



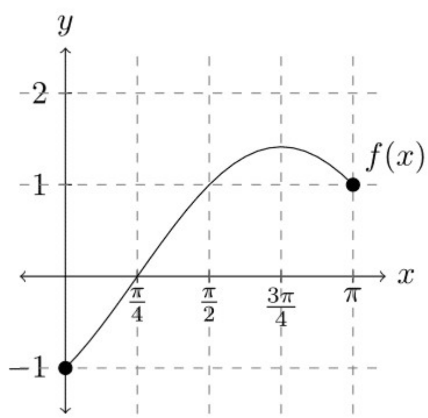
12.

a. $f'(x) = \cos x + \sin x$; $f''(x) = -\sin x + \cos x$

b. Critical point $x = \frac{3\pi}{4}$; $f\left(\frac{3\pi}{4}\right) = \sqrt{2}$, $f(0) = -1$, $f(\pi) = 1$. Global maximum: $\left(\frac{3\pi}{4}, \sqrt{2}\right)$; global minimum: $(0, -1)$.

c. Inflection $\left(\frac{\pi}{4}, 0\right)$. Concave up on $\left(0, \frac{\pi}{4}\right)$, concave down on $\left(\frac{\pi}{4}, \pi\right)$.

d. Sketch



13. Use $f(x + \Delta x) \approx f(x) + \Delta x \cdot f'(x)$. $f(120) = f(121 - 1)$. Answer: $10\frac{21}{22}$.

14. $e^{0.1} \approx 1.1$

15. $f(x)$ is a polynomial and thus continuous and differentiable for all x. $c = 1$.

16. Average rate of change from $(0, 0°\text{F})$ to $(16, 212°\text{F})$ is $13.25°\text{F}$. MVT says there is a time between $t = 0$ and $t = 16$ s where the instantaneous rate of change is $13.25°\text{F}$.

17. Average velocity after 2 s is -9.8 m/s. Instantaneous velocity is -19.6 m/s.

18. $\frac{dh}{dt}$ @ $h = 10 \text{ ft} = \frac{6}{5\pi}$ ft/min

19. $\frac{630}{13}$ ft/s

20. Velocity = -12 ft/s. Speed (how fast) = 12 ft/s.

21. $\frac{1}{2\pi} ft/min$

22. $\frac{80}{17} rad/hr$ OR $\frac{4}{51} rad/min$

23. $r = \sqrt[3]{\frac{1}{60\pi}}\ meters$

24. base of 18 in and a height of 36 in

25. base of $\sqrt{3}$ and height of $\frac{3}{2}$.

26. a. 60 ft

b. 10 ft x 15 ft

27. a. i. -9 m/s

ii. $v(t) = -6t + 6, a(t) = -6$
iii. $v(4) = -18\, m/s$
iv. Positive direction $(0,1)$, negative direction $(1, \infty)$.
v. Speeding up (1,4) because the velocity and acceleration have the same sign.
Slowing down $(0,1)$ because velocity and acceleration have opposite signs.

b. i. 0 m/s

ii. $v(t) = 3t^2 - 12t + 9, a(t) = 6t - 12$
iii. 9 m/s
iv. Positive direction $(0,1)$ and $(3, \infty)$, negative direction $(1,3)$
v. Speeding up $(1,2)$ and $(3,4)$ because the velocity and acceleration have the same sign.
Slowing down $(0,1)$ and $(2,3)$ because velocity and acceleration have opposite signs.

28. a. $F(x) = \frac{1}{2}x^4 - \frac{1}{3}e^x + 3x + ex + C$

b. $G(x) = \frac{7}{6}x^6 + x^2 + \ln x - e^x + C$